

## SOME NEW INEQUALITIES FOR THE $q$ -GAMMA AND RELATED FUNCTIONS

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ABSTRACT. We consider convexity and monotonicity properties for some functions related to the  $q$ -gamma function. As applications, we give a variety of inequalities for the  $q$ -gamma function, the  $q$ -digamma function  $\psi_q(x)$ , and the  $q$ -series. Among other consequences, we improve a result of Alzer and Grinshpan about the zeros of the function  $\psi_q(x)$ . We use  $q$ -analogues for the Gauss multiplication formula to put in closed form members of some of our inequalities.

### 1. INTRODUCTION

Throughout this paper we assume that  $0 < q < 1$ . The  $q$ -shifted factorials of a complex number  $a$  are defined by

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i), \quad (a; q)_\infty = \lim_{n \rightarrow \infty} (a; q)_n.$$

For convenience we write

$$(a_1, \dots, a_k; q)_n = (a_1; q)_n \cdots (a_k; q)_n, \quad (a_1, \dots, a_k; q)_\infty = (a_1; q)_\infty \cdots (a_k; q)_\infty.$$

For any complex  $x$ , we let

$$[x]_q = \frac{1 - q^x}{1 - q},$$

for which we have  $\lim_{q \rightarrow 1} [x]_q = x$ . The  $q$ -gamma function is given by

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x} \quad (|q| < 1).$$

It is clear that

$$(1) \quad \Gamma_q(x + 1) = [x]_q \Gamma_q(x)$$

and it is well-known that  $\Gamma_q(x)$  is a  $q$ -analogue for the function  $\Gamma(x)$ , see Askey [3]. The digamma function is  $\psi(x) = (\log \Gamma(x))' = \frac{\Gamma'(x)}{\Gamma(x)}$  and its  $q$ -analogue is the  $q$ -digamma function given by

$$\psi_q(x) = (\log \Gamma_q(x))' = \frac{\Gamma'_q(x)}{\Gamma_q(x)}.$$

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The  $q$ -binomial theorem states that

$$(2) \quad \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} x^n = \frac{(ax; q)_{\infty}}{(x; q)_{\infty}} =: {}_1\phi_0(a, -; q, x) \quad (|x| < 1, |q| < 1),$$

where  ${}_1\phi_0(a, -; q, x)$  is the basic hypergeometric series. For details and historical notes on the  $q$ -series, the hypergeometric series, and related functions we refer to [1]. Our primary goal in this paper is to consider monotonicity and convexity properties of the  $q$ -gamma function and some of its related functions. As an application, we shall present inequalities involving the functions  $\Gamma_q(x)$  and  $\psi_q(x)$  along with related functions including the function  ${}_1\phi_0(a, -; q, x)$ . Some of our inequalities involve powers, ratios, and products of these special functions. A crucial tool to achieve some of our inequalities is Jensen’s inequality stating that if  $f(x)$  is a convex function on  $I$  then for all  $x_1, \dots, x_n \in I$  and all positive  $a_1, \dots, a_n$  one has

$$f\left(\frac{\sum_{i=1}^n a_i x_i}{\sum_{i=1}^n a_i}\right) \leq \frac{\sum_{i=1}^n a_i f(x_i)}{\sum_{i=1}^n a_i}.$$

We mention that refinements of Jensen’s inequality exist in literature and thus any inequality we prove in this paper using Jensen’s inequality can appropriately be slightly improved. For some of other refinements of Jensen’s inequality, see [16]. Alzer and Grinshpan [2, Lemma 4.5] proved that the function  $\psi_q(x)$  for  $0 < q \neq 1$  has a uniquely determined positive zero  $x_0 = x_0(q)$ . Among our applications, we shall show that  $x_0(q) \in (1, 2)$ . We will also provide Ky Fan type inequalities for the  $q$ -gamma function. Another purpose of our work is to establish a variety of inequalities involving the  $q$ -series. We note that some of our formulas have been put in closed forms thanks to  $q$ -analogues of the Gauss multiplication formula for the gamma function which we shall describe now. We recall that the Gaussian multiplication formula for gamma functions states that

$$(3) \quad \Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{2}{n}\right) \cdots \Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}} \quad (n = 1, 2, \dots).$$

A famous  $q$ -analogue for (3) due to Jackson [9, 10], states that

$$(4) \quad \begin{aligned} & \left(\frac{1-q^n}{1-q}\right)^{nz-1} \Gamma_{q^n}(z) \Gamma_{q^n}\left(z + \frac{1}{n}\right) \cdots \Gamma_{q^n}\left(z + \frac{n-1}{n}\right) \\ &= \Gamma_{q^n}(nz) \Gamma_{q^n}\left(\frac{1}{n}\right) \Gamma_{q^n}\left(\frac{2}{n}\right) \cdots \Gamma_{q^n}\left(\frac{n-1}{n}\right) \quad (n = 1, 2, \dots). \end{aligned}$$

Recently, the authors [4] gave the following  $q$ -analogue for (3)

$$(5) \quad \prod_{k=1}^{n-1} \Gamma_q\left(\frac{k}{n}\right) = \left(\Gamma_q\left(\frac{1}{2}\right)\right)^{n-1} \frac{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{\infty}^{n-1}}{(q; q)_{\infty}^{n-2} (q^{\frac{1}{n}}; q^{\frac{1}{n}})_{\infty}}.$$

Besides, Sándor and Tóth [21] found

$$(6) \quad P(n) := \prod_{\substack{k=1 \\ (k,n)=1}}^n \Gamma\left(\frac{k}{n}\right) = \frac{(2\pi)^{\frac{\varphi(n)}{2}}}{\prod_{d|n} d^{\frac{1}{2}\mu\left(\frac{n}{d}\right)}} = \frac{(2\pi)^{\frac{\varphi(n)}{2}}}{e^{\frac{\Lambda(n)}{2}}},$$

where  $\varphi(n)$  is the Euler totient function,  $\mu(n)$  is the Möbius mu function, and  $\Lambda(n)$  is the Von Mangoldt function. We accordingly let

$$P_q(n) = \prod_{\substack{k=1 \\ (k,n)=1}}^n \Gamma_q\left(\frac{k}{n}\right).$$

The authors [4] also found the following  $q$ -analogue (6).

$$(7) \quad P_q(n) = \frac{\left(\Gamma_q\left(\frac{1}{2}\right)\right)^{\varphi(n)} (q^{\frac{1}{2}}; q)^{\varphi(n)}}{\prod_{d|n} \left((q^{\frac{1}{d}}; q^{\frac{1}{d}})_{\infty}\right)^{\mu\left(\frac{n}{d}\right)}} = \frac{\left(2\Gamma_q\left(\frac{1}{2}\right)\right)^{\frac{\varphi(n)}{2}}}{e^{\frac{\Lambda_q(n)}{2}}}.$$

### 2. INEQUALITIES FOR $\psi_q(x)$ AND $\Gamma_q(x)$

**Lemma 1.** (a) *The derivative of the function  $\psi_q(x)$  is strictly completely monotonic on  $(0, \infty)$ , that is,*

$$(-1)^n (\psi'_q(x))^{(n)} > 0 \quad x > 0, \quad n = 0, 1, 2, \dots$$

(b) *For any  $x \geq 1$ , we have that  $x(\psi_q(x))' + 2\psi_q(x) > 0$ .*

*Proof.* Part (a) is an immediate consequence of the series representation

$$\psi_q(x) = -\log(1 - q) + (\log q) \sum_{n=0}^{\infty} \frac{q^{n+x}}{1 - q^{n+x}} = -\log(1 - q) + (\log q) \sum_{n=1}^{\infty} \frac{q^{nx}}{1 - q^n}.$$

Part (b) is due Alzer and Grinshpan [2, Lemma 3.4]. □

**Lemma 2.** *For all  $x > 0$ , we have*

$$\frac{q^x \log q}{1 - q^x} + \log[x]_q < \psi_q(x) < \log[x]_q.$$

*Proof.* From (1), we deduce that  $\log \Gamma_q(x + 1) - \log \Gamma_q(x) = \log[x]_q$ . Then by Lagrange mean value theorem, there exists  $t \in (0, 1)$  such that

$$(8) \quad \psi_q(x + t) = \log[x]_q.$$

As  $\psi_q(x)$  is strictly increasing by Lemma 1, the forgoing identity implies that

$$(9) \quad \psi_q(x) < \psi_q(x + t) < \psi_q(x + 1).$$

Next, differentiate both sides of (1) to obtain

$$(10) \quad \psi_q(x) = \frac{q^x \log q}{1 - q^x} + \psi_q(x + 1).$$

Now, combine (8), (9), and (10) to get the desired inequalities. □

Alzer and Grinshpan [2, Lemma 4.5] proved that the function  $\psi_q(x)$  for  $0 < q \neq 1$  has a uniquely determined positive zero  $x_0 = x_0(q)$ . For  $0 < q < 1$ , it turns out that  $x_0(q) \in (0, 1)$  as we will see now.

**Theorem 1.** (a) *The function  $\psi_q(x)$  has a unique zero  $x_0$  in the interval  $(1, 2)$ .*  
 (b) *There holds  $\Gamma_q(x) \geq \Gamma_q(x_0)$  for all  $x \in (0, \infty)$ .*

*Proof.* First proof of (a) Application of Lemma 2 to  $x = 1$  and to  $x = 2$  respectively gives

$$(11) \quad \frac{q \log q}{1 - q} < \psi_q(1) < 0 \quad \text{and} \quad \frac{q^2 \log q}{1 - q^2} + \log(1 + q) < \psi_q(2).$$

By the well-known fact that the function  $\psi(x)$  is strictly increasing and continuous we will be done if we show that

$$(12) \quad \frac{q^2 \log q}{1 - q^2} + \log(1 + q) > 0.$$

Letting  $q = \frac{1}{t}$  for  $t > 1$  and after simplification (12) becomes

$$\log t < \log(t + 1) - \frac{\log t}{t^2 - 1},$$

or equivalently,

$$(t^2 - 1) \log(t + 1) - t^2 \log t > 0.$$

Letting  $f(t) = (t^2 - 1) \log(t + 1) - t^2 \log t$ , we find that

$$\begin{aligned} f'(t) &= 2t \log(t + 1) - 2t \log t - 1 \\ f''(t) &= 2 \left( \log \left( 1 + \frac{1}{t} \right) - \frac{1}{t + 1} \right) \end{aligned}$$

Then by a combination of the previous identity and the well-known inequality  $\left(1 + \frac{1}{t}\right)^{t+1} > e$ , we deduce that  $f''(t) > 0$  from which it follows that  $f'(t)$  is strictly increasing. Then from the above,  $f'(t) > f'(1) = 2 \log 2 - 1 > 0$ , which in turn shows that  $f(t)$  is strictly increasing. Therefore  $f(t) > f(1) = 0$ , establishing the relation (12).

*Second proof of (a)* As  $\Gamma_q(1) = \Gamma_q(2) = 1$ , we have by Rolle's theorem applied to  $\Gamma_q(x)$  on  $[1, 2]$  there exists  $x_0 \in (1, 2)$  such that  $(\Gamma_q(x_0))' = 0$  and hence  $\psi_q(x_0) = 0$  as  $(\Gamma_q(x))' = \psi_q(x)\Gamma_q(x)$ . Since  $\Gamma_q(x)$  is strictly convex, its derivative is strictly increasing, and so  $x_0$  is unique.

(b) It is well-known that a strict log-convex function is also strict convex and so,  $\Gamma_q(x)$  is strict convex on  $(0, \infty)$  by Lemma 1(a). That is,  $(\Gamma_q(x))'$  is strictly increasing on  $(0, \infty)$ . Now combine this with the identity  $(\Gamma_q(x))' = \psi_q(x)\Gamma_q(x)$  as follows. Then  $(\Gamma_q(x))' < (\Gamma_q(x_0))' = 0$  on the left of  $x_0$  and  $(\Gamma_q(x))' \geq (\Gamma_q(x_0))' = 0$  on the right of  $x_0$ , showing that the function  $\Gamma_q(x)$  is strictly decreasing on  $(0, x_0)$  and strictly increasing on  $(x_0, \infty)$ . This completes the proof.  $\square$

**Remark 1.** By the known inequality  $\log x < x - 1$  for  $1 \neq x > 0$  applied to  $x = \frac{1}{q}$ , we get  $\frac{q \log q}{q} > -1$  and so by (11), we get

$$(13) \quad -1 < \psi_q(1) < 0.$$

Bradley [6] introduced an extension of the Euler gamma constant  $\gamma$  as follows

$$\gamma_q = \log(1 - q) - \frac{\log q}{1 - q} \sum_{i=1}^{\infty} \frac{q^i}{[i]_q}$$

and proved that  $\lim_{q \rightarrow 1} \gamma_q = \gamma$ . Mahmoud and Agarwal [12] proved that for  $0 < q < 1$ , we have  $\psi_q(1) = \gamma_q$ . We note that there is no any other information in [6]

and [12] related to the generalized constant  $\gamma_q$ . From (13), it follows that

$$0 < \gamma_q < 1 \text{ for any } q \in (0, 1).$$

Stronger approximations are given in [5].

**Theorem 2.** (a) *The function  $\log \Gamma_q(x) + x\psi_q(x)$  is strictly increasing on  $(1, \infty)$  with a single zero which is in  $(1, 2)$ .*

(b) *The function  $\log \Gamma_q(x) - x\psi_q(x)$  is strictly decreasing on  $(0, \infty)$  with a single zero which is in  $(1, 2)$ .*

*Proof.* Let  $f(x) = \log \Gamma_q(x) + x\psi_q(x)$  for any  $x \in (0, \infty)$ . Then by Lemma 1(b),  $f'(x) = 2\psi_q(x) + x\psi_q'(x) > 0$  and therefore  $f(x)$  is strictly increasing on  $(1, \infty)$ . We have already seen in the proof of Theorem 1 that  $\psi_q(1) < 0 < \psi_q(2)$ . It follows that  $f(1) = \psi_q(1) < 0 < 2\psi_q(2) = f(2)$ . As the function  $f(x)$  is clearly continuous on  $(1, \infty)$ , the proof is complete for part (a). Part (b) follows in exactly the same way.  $\square$

**Corollary 1.** *For any  $x > 1$  and any positive integer  $n$  we have the following double inequality*

$$x\psi_q(x) - (x+n)\psi_q(x+n) < \log \frac{\Gamma_q(x)}{\Gamma_q(x+n)} < (x+n)\psi_q(x+n) - x\psi_q(x).$$

*Proof.* By Theorem 2(a), we have  $\log \Gamma_q(x) + x\psi_q(x) < \log \Gamma_q(y) + y\psi_q(y)$  whenever  $1 < x < y$ . Repeatedly application of this and simplifying yield

$$\begin{aligned} \log \Gamma_q(x) - \log \Gamma_q(x+1) &< (x+1)\psi_q(x+1) - x\psi_q(x) \\ \log \Gamma_q(x+1) - \log \Gamma_q(x+2) &< (x+2)\psi_q(x+2) - (x+1)\psi_q(x+1) \end{aligned}$$

$\vdots$

$$\log \Gamma_q(x+n-1) - \log \Gamma_q(x+n) < (x+n)\psi_q(x+n) - (x+n-1)\psi_q(x+n-1).$$

Adding together gives

$$\log \Gamma_q(x) - \log \Gamma_q(x+n) < -x\psi_q(x) + (x+n)\psi_q(x+n),$$

which is equivalent to the first inequality. The second inequality is obtained similarly by considering the function  $\log \Gamma_q(x) - x\psi_q(x)$  which is decreasing by Theorem 2(a).  $\square$

**Remark 2.** It is known, see for instance [3], that the function  $x\psi_q(x)$  is strictly convex, so, as  $\log \Gamma_q(x)$  is strictly convex too, we get that the function  $f(x) = \log \Gamma_q(x) + x\psi_q(x)$  of Theorem 2(a) is strictly convex.

**Lemma 3.** *For any positive integers  $k$  and  $n$  there holds*

$$\begin{aligned} (a) \quad \sum_{i=1}^{n-1} \psi_q\left(\frac{i}{n}\right) &= (n-1)\psi_q(1) - n \log \frac{1-q}{1-q^{\frac{1}{n}}} \\ (b) \quad \sum_{i=1}^{n-1} \psi_q^{(k)}\left(\frac{i}{n}\right) &= (n^{k+1}-1)\psi_q^{(k)}(1). \end{aligned}$$

*Proof.* In (4) replacing  $q^n$  with  $q$  and taking logarithms on both sides give

$$(14) \quad (nz-1) \log \frac{1-q}{1-q^{\frac{1}{n}}} + \sum_{i=0}^{n-1} \log \Gamma_q\left(z + \frac{i}{n}\right) = \log \Gamma_q(nz) + \log \prod_{i=1}^{n-1} \Gamma_q\left(\frac{i}{n}\right).$$

Differentiating with respect to  $z$  and then letting  $z = \frac{1}{n}$  yield

$$n \log \frac{1-q}{1-q^{\frac{1}{n}}} + \sum_{i=1}^{n-1} \psi_q\left(\frac{i}{n}\right) + \psi_q(1) = n\psi_q(1),$$

which is equivalent to the desired identity in part (a). To prove part (b), first differentiate with respect  $z$ ,  $k$  times both sides of (14) to obtain

$$\sum_{i=0}^{n-1} \psi_q^{(k)}\left(z + \frac{i}{n}\right) = n^{k+1} \psi_q^{(k)}(nz),$$

then let  $z = \frac{1}{n}$  to get

$$\sum_{i=1}^{n-1} \psi_q^{(k)}\left(\frac{i}{n}\right) = (n^{k+1} - 1) \psi_q^{(k)}(1),$$

as desired. □

**Theorem 3.** *For any positive integers  $k$  and  $n$  there holds*

- (a)  $(n - 1) \left( \psi_q(1) - \psi_q\left(\frac{1}{2}\right) \right) < n \log \frac{1-q}{1-q^{\frac{1}{n}}}$
- (b)  $(n - 1) \psi_q^{(2k-1)}\left(\frac{1}{2}\right) < (n^{2k} - 1) \psi_q^{(2k-1)}(1)$
- (c)  $(n - 1) \psi_q^{(2k)}\left(\frac{1}{2}\right) > (n^{2k+1} - 1) \psi_q^{(2k)}(1).$

*Proof.* (a) By Lemma 1(a), the function  $\psi_q(x)$  is strictly concave. Then by an application of Jensen’s inequality to this function with  $k = n - 1$  and  $x_i = \frac{i}{n}$  for  $i = 1, \dots, n - 1$  we find

$$\psi_q\left(\frac{\sum_{i=1}^{n-1} \frac{i}{n}}{n-1}\right) > \frac{1}{n-1} \sum_{i=1}^{n-1} \psi_q\left(\frac{i}{n}\right).$$

Then by an appeal to Lemma 3(a) along with simplification we derive

$$(n - 1) \psi_q\left(\frac{1}{2}\right) > (n - 1) \psi_q(1) - n \log \frac{1-q}{1-q^{\frac{1}{n}}},$$

which proves part (a).

(b) By Lemma 1(a), the function  $\psi_q^{(2k-1)}(x)$  is strictly convex and therefore by Jensen’s inequality applied to this function with  $k = n - 1$  and  $x_i = \frac{i}{n}$  for  $i = 1, \dots, n - 1$  one has

$$\psi_q^{(2k-1)}\left(\frac{\sum_{i=1}^{n-1} \frac{i}{n}}{n-1}\right) < \frac{1}{n-1} \sum_{i=1}^{n-1} \psi_q^{(2k-1)}\left(\frac{i}{n}\right).$$

Now use Lemma 3(b) and simplify to deduce that

$$(n - 1) \psi_q^{(2k-1)}\left(\frac{1}{2}\right) < (n^{2k} - 1) \psi_q^{(2k-1)}(1),$$

which is the desired relation in part (b). The similar proof of part (c) is omitted. □

3. CONVEXITY AND INEQUALITIES FOR POWERS, RATIOS, AND PRODUCTS OF  $\Gamma_q(x)$

**Lemma 4.** *Let  $f : (0, \infty) \rightarrow (0, \infty)$  and let  $g(x) = \frac{f(x+1)}{f(x)}$ . If  $f(x)$  is strictly log-convex on  $(0, \infty)$ , then for any  $x > 0$  and any  $a \in (0, 1)$  we have*

$$(g(x))^{1-a} < \frac{f(x+1)}{f(x+a)} < (g(x+a))^{1-a}.$$

*Proof.* Since  $f(x)$  is strictly log-convex on  $(0, \infty)$ , we have for any  $u \in [0, 1]$  and any  $y \neq z > 0$

$$\log(f(uy + (1-u)z)) < u \log f(y) + (1-u) \log f(z)$$

or equivalently,

$$(15) \quad f(uy + (1-u)z) < (f(y))^u (f(z))^{1-u}.$$

Let in (15)  $y := x$ ,  $z := x + 1$ , and  $u := 1 - a$  to obtain

$$f(x+a) < (f(x))^{1-a} (f(x+1))^a,$$

from which we easily get the first inequality. As to the second inequality, let in (15)  $y := x + a$ ,  $z := x + a + 1$ , and  $u := a$  and proceed as before.  $\square$

A classical result by Gautschi [8] states that

$$x^{1-a} < \frac{\Gamma(x+1)}{\Gamma(x+a)} < e^{(1-a)\psi(x+1)} \quad (0 < a < 1).$$

We have the following  $q$ -variant which seems to be new.

**Corollary 2.** *Let  $x > 0$  and  $a \in (0, 1)$ . Then*

$$([x]_q)^{1-a} < \frac{\Gamma_q(x+1)}{\Gamma_q(x+a)} < ([x+a]_q)^{1-a}.$$

*Proof.* Simply apply Lemma 4 to the function  $f(x) = \log \Gamma_q(x)$ .  $\square$

The following result is well-known.

**Lemma 5.** *Let  $I \subseteq \mathbb{R}$  and let  $f : I \rightarrow (0, \infty)$ .*

*(a) If  $f(x)$  is concave (strictly concave), then  $\frac{1}{f(x)}$  is convex (strictly convex).*

*(b) The function  $f(x)$  is log-convex if and only if  $\frac{1}{f(x)}$  is log-concave.*

Note that the converse of Lemma 5(a) is not true. For example, the function  $e^x$  is convex but the reciprocal  $e^{-x}$  is not concave.

**Lemma 6.** *For any  $x > 0$  we have*

$$(a) \quad (\psi_q(x+1))' < \frac{-q^x \log q}{1-q^x} < (\psi_q(x))'$$

$$(b) \quad (\psi_q(x))'' < \frac{-q^x (\log q)^2}{(1-q^x)^2} < (\psi_q(x+1))''.$$

*Proof.* By the Lagrange mean value theorem there is  $y \in (x, x+1)$  such that  $\psi_q(x+1) - \psi_q(x) = (\psi_q(y))'$ . As the the function  $(\psi_q(x))'$  is strictly decreasing by Lemma 1(a), a combination of the previous identity with the relation (10) yields part (a). To establish part (b), note first that using the Lagrange mean value

theorem there exists  $z \in (x, x + 1)$  such that  $\psi_q(x + 1) - \psi_q(x) = (\psi_q(z))''$ . Note also that by (10) one has

$$(\psi_q(x + 1))' - (\psi_q(x))' = \frac{-q^x (\log q)^2}{(1 - q^x)^2}.$$

Moreover, the function  $(\psi_q(x))'$  is strictly increasing by Lemma 1(a). Thus part (b) follows by a combination of the above facts.  $\square$

**Corollary 3.** *There holds*

$$\begin{aligned} \text{(a)} \quad & (\psi_q(x + 1))' < \frac{1}{x}, \text{ for all } x > 0 \\ \text{(b)} \quad & (\psi_q(x + 1))'' > \frac{-1}{x^2}, \text{ for all } x > 0. \end{aligned}$$

*Proof.* Upon letting  $t = q^x$  and using Lemma 6, we see that to prove part (a) it will be enough to show that  $\frac{t \log t}{1-t} > -1$ . But this inequality has been established in Remark 1. Similarly, but now letting  $q^x = \frac{1}{p}$  for  $p > 1$ , we can check that in order to prove part (b), it will be enough to show that

$$\sqrt{p} < \frac{p-1}{\log p}.$$

To this end, let  $A(a, b)$  be the arithmetic mean,  $G(a, b)$  be the geometric mean, and  $L(a, b)$  be the logarithmic mean, i.e.

$$A(a, b) = \frac{a+b}{2}, \quad G(a, b) = \sqrt{ab}, \quad L(a, b) = \frac{b-a}{\log b - \log a}.$$

It is well-known (see for instance Sándor [17, 20]) for these means that

$$(16) \quad G(a, b) < L(a, b) < A(a, b).$$

In particular, we have that  $G(p, 1) < L(p, 1)$ , which is the desired inequality.  $\square$

**Theorem 4.** *The function  $f(x) = (\Gamma_q(x + 1))^{\frac{1}{x}}$  is strictly log-concave and strictly increasing on  $(0, \infty)$ .*

*Proof.* We find

$$(17) \quad x^3 (\log f(x))'' = x^2 (\psi_q(x + 1))' - 2x \psi_q(x + 1) + 2 \log \Gamma_q(x + 1)$$

and letting  $h(x) = x^2 (\psi_q(x + 1))' - 2x \psi_q(x + 1) + 2 \log \Gamma_q(x + 1)$ , we have  $h'(x) = x^2 (\psi_q(x + 1))''$  and so  $h'(x) < 0$  by Lemma 1(a), that is,  $h(x)$  is strictly decreasing on  $(0, \infty)$ . Then  $h(x) < h(0) = 0$ , which combined with equation (17) implies that  $\log f(x)$  is strictly concave on  $(0, \infty)$  and therefore, the first statement follows. As to the monotonicity, observe that  $(\log f(x))' = \frac{a(x)}{x^2}$ , where

$$a(x) = x \psi_q(x + 1) - \log \Gamma_q(x + 1).$$

The function  $a(x)$  is well-defined on  $[0, \infty)$ . One clearly has  $a(0) = 0$  and by Lemma 1,  $a'(x) = x (\psi_q(x))' > 0$ . Thus  $a(x) > a(0) = 0$  and we have that the function  $f(x)$  is strictly increasing on  $(0, \infty)$ .  $\square$

**Corollary 4.** *Let  $f(x)$  be the function in Theorem 4. Then for any  $x > 0$  we have*

$$e^{-\gamma_q} < f(x) < \frac{1}{1-q}.$$

*Proof.* Since  $f(x)$  is strictly increasing by Theorem 4, one has

$$f(0^+) < f(x) < f(\infty).$$

Recall further that  $\psi_q(1) = -\gamma_q$  and note that by Alzer and Grinshpan [2] we have  $\lim_{x \rightarrow \infty} \psi_q(x) = -\log(1 - q)$ . Now, by l'Hopital's rule we find that

$$\lim_{x \rightarrow \infty} \log f(x) = \psi_q(1) = -\gamma_q \text{ and } \lim_{x \rightarrow \infty} \log f(x) = \lim_{x \rightarrow \infty} \psi_q(x + 1) = -\log(1 - q),$$

which yields the desired inequalities.  $\square$

**Corollary 5.** *The function  $\frac{1}{\Gamma_q(x+1)^{\frac{1}{x}}}$  is strictly log-convex on  $(0, \infty)$ .*

*Proof.* This follows by Theorem 4 and Lemma 5(b).  $\square$

A classical gamma version for Corollary 5 is due to Van de Lune [23].

**Corollary 6.** *Let*

$$g(x) = \left( \frac{\left( \Gamma_q(x + 1) \right)^{\frac{1}{x}}}{[x + 1]_q} \right)^{\frac{1}{x+1}}.$$

*Then for any  $x > 0$  and any  $a \in (0, 1)$ , we have*

$$(g(x))^{1-a} < \frac{\left( \Gamma_q(x + a + 1) \right)^{\frac{1}{x+a}}}{\left( \Gamma_q(x + 2) \right)^{\frac{1}{x+1}}} < (g(x + a))^{1-a}.$$

*Proof.* This is a direct consequence of Corollary 5 and Lemma 2 and the basic fact that  $\Gamma_q(x + 2) = [x + 1]_q \Gamma_q(x + 1)$ .  $\square$

**Theorem 5.** *Let  $f(x)$  be as defined in Theorem 4. Then the function  $F(x) = \frac{f(x)}{x}$  is strictly decreasing and strictly log-convex on  $(0, \infty)$ .*

*Proof.* Let  $b(x) = \log F(x)$  and  $c(x) = x^2 b'(x)$ . Then we easily find that  $b(x) = -\log x + \frac{1}{x} \log \Gamma_q(x + 1)$  and therefore that  $c(x) = -x - \log \Gamma_q(x + 1) + x \psi_q(x + 1)$ . Note that the function  $c(x)$  can be defined for all  $x \geq 0$  and we have that  $c(0) = 0$  and  $c'(x) = -1 + x(\psi_q(x + 1))'$ . The last identity and Corollary 3(a) imply that  $c'(x) < 0$ , thus  $c(x) < c(0)$ , so  $b'(x) < 0$ , and consequently the function  $b(x)$  is strictly decreasing. Now we consider the convexity of  $b(x)$ . As we have seen above, one has

$$b'(x) = \frac{-1}{x} - \frac{1}{x^2} \log \Gamma_q(1 + x) + \frac{1}{x} \psi_q(1 + x),$$

from where we obtain, by letting  $d(x) = x^3 b''(x)$ , that

$$d(x) = x + 2 \log \Gamma_q(x + 1) - 2x \psi_q(x + 1) + x^2 (\psi_q(x + 1))'$$

The function can be defined on  $[0, \infty)$ . One has  $d(0) = 0$  and we obtain after some computations that  $d'(x) = 1 + x^2 (\psi_q(x + 1))''$ . Then by virtue of Corollary 3(b) we have that  $d'(x) > 0$ . Thus  $d(x) > d(0) = 0$ , and the result follows.  $\square$

**Remark 3.** The two monotonicity properties of the functions  $f(x)$  of Theorem 4 and of  $\frac{f(x)}{x}$  of Theorem 5 have been proved for the case of the classical gamma function by Kershaw and Laforgia [11]. We note also that in [11] it was proved that the function  $\Gamma\left(1 + \frac{1}{x}\right)$  is strictly decreasing and  $x\Gamma\left(1 + \frac{1}{x}\right)^x$  is strictly increasing.

It is immediate that these are equivalent with the above monotonicity theorems for  $q = 1$ .

**Theorem 6.** (a) The function  $(\Gamma_q(x))^{\frac{1}{x}}$  is strictly log-convex on  $(0, 1]$ .

(b) The function  $(\Gamma_q(x))^x$  is strictly log-convex on  $[1, \infty)$ .

*Proof.* (a) Letting  $f(x) = (\Gamma_q(x))^{\frac{1}{x}}$ , we get

$$(18) \quad x^3(\log f(x))'' = x^2(\psi_q(x))' - 2x\psi_q(x) + 2\log \Gamma_q(x+1).$$

Now for the function  $h(x) = x^2(\psi_q(x))' - 2x\psi_q(x) + 2\log \Gamma_q(x+1)$  we find  $h'(x) = x^2(\psi_q(x))'' < 0$ , that is, the function  $h(x)$  decreases on  $(0, 1]$ . Then for any  $x \in (0, 1]$  we have with the help of Lemma 1(a) and inequality (11)

$$h(x) \geq h(1) = \psi_q'(1) - 2\psi_q(1) > 0.$$

Thus  $(\log f(x))'' > 0$  on  $(0, 1]$ , or equivalently,  $f(x) = (\Gamma_q(x))^{\frac{1}{x}}$  is log-convex on  $(0, 1]$ . (b) We have by a straight computation and Lemma 1(b),

$$(\log (\Gamma_q(x))^x)'' = 2\psi_q(x) + x(\psi_q(x))' > 0,$$

showing the desired statement.  $\square$

We note that Theorem 4 and Theorem 6(a) were motivated by results of the second author in [18, 19] on Euler gamma function.

**Lemma 7.** Let  $f(x)$  be strictly log-convex on the interval  $(0, 1)$ . Then we have

$$\frac{f(1-2x(1-x))}{f(1-x)} < \left(\frac{f(x)}{f(1-x)}\right)^x < \frac{f(x)}{f(2x(1-x))}.$$

*Proof.* As  $f(x)$  is strictly log-convex we have for any  $a \in (0, 1)$

$$f(a(1-x) + (a-1)x) < (f(1-x))^a (f(x))^{1-a},$$

which by letting  $a = 1-x$  means

$$f(2x^2 - 2x + 1) < (f(1-x))^{1-x} (f(x))^x.$$

It follows that

$$\frac{f(1-2x(1-x))}{f(1-x)} < \left(\frac{f(x)}{f(1-x)}\right)^x,$$

which is the first desired inequality. As to the second inequality, as  $f(x)$  is strictly log-convex we also have for any  $a \in (0, 1)$

$$f(ax + (a-1)(1-x)) < (f(x))^a (f(1-x))^{1-a},$$

which by letting  $a = 1-x$  means

$$f(2x(1-x)) < (f(x))^{1-x} (f(1-x))^x,$$

or equivalently,

$$\frac{f(x)}{f(2x(1-x))} > \left(\frac{f(x)}{f(1-x)}\right)^x.$$

This completes the proof.  $\square$

**Corollary 7.** For any  $x \in (0, 1)$  we have

- (a)  $\Gamma_q(2x(1-x))\Gamma_q(1-2x(1-x)) < \Gamma_q(x)\Gamma_q(1-x)$
- (b)  $\psi_q(2x(1-x))\psi_q(1-2x(1-x)) > \psi_q(x)\psi_q(1-x)$ .

*Proof.* Let  $f(x) = \Gamma_q(x)\Gamma_q(1-x)$ . Then  $f(x)$  is strictly log-convex, being the product of two log-convex functions and we clearly have  $f(x) = f(1-x)$ . Then by virtue of Lemma 7, we get

$$\frac{\Gamma_q(2x(1-x))\Gamma_q(1-2x(1-x))}{\Gamma_q(x)\Gamma_q(1-x)} < 1 < \frac{\Gamma_q(x)\Gamma_q(1-x)}{\Gamma_q(2x(1-x))\Gamma_q(1-2x(1-x))}.$$

Then simplifying gives part (a). As to part (b), let  $g(x) = \psi_q(x)\psi_q(1-x)$ . Then by Lemma 1(a) we find

$$(\log g(x))'' = (\psi(x))'' + (\psi(1-x))'' < 0,$$

showing by Lemma 5 that the reciprocal  $\frac{1}{g(x)}$  is strictly log-convex. Moreover, it is clear that  $g(x) = g(1-x)$ . Then from Lemma 7, we have

$$\frac{\psi_q(x)\psi_q(1-x)}{\psi_q(2x(1-x))\psi_q(1-2x(1-x))} < 1.$$

This completes the proof. □

For our next result, we need the following result of Vasić [22] which is an extension of a famous inequality of Petrović. We refer to [13] for details about Petrović's inequality.

**Lemma 8.** Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be convex. Then for any  $x_1, \dots, x_n \geq 0$  and any  $p_1, \dots, p_n \geq 1$ , we have

$$\sum_{i=1}^n p_i f(x_i) \leq f\left(\sum_{i=1}^n p_i x_i\right) + \left(\sum_{i=1}^{n-1} p_i - 1\right) f(0).$$

**Corollary 8.** For any real numbers  $x_1, \dots, x_n \geq 0$ , we have

$$\prod_{i=1}^n \Gamma_q(x_i) \leq \frac{[\sum_{i=1}^n x_i]_q}{\prod_{i=1}^n [x_i]_q} \Gamma_q\left(\sum_{i=1}^n x_i\right).$$

*Proof.* Simply apply Lemma 8 to  $f(x) = \log \Gamma_q(x+1)$ ,  $p_1 = \dots = p_n = 1$  and note that  $f(0) = 0$ . □

#### 4. INEQUALITIES RELATED TO $\frac{\Gamma_q(1-x)}{\Gamma_q(x)}$ AND $\Gamma_q\left(\frac{1-x}{x}\right)$

**Theorem 7.** (a) The function  $\Gamma_q\left(\frac{1-x}{x}\right)$  is strictly log-convex on  $(0, \frac{1}{2}]$ .

(b) The function  $\frac{\Gamma_q(1-x)}{\Gamma_q(x)}$  is strictly log-concave on  $(0, \frac{1}{2}]$ .

*Proof.* (a) Let  $f_1(x) = \log \Gamma_q\left(\frac{1-x}{x}\right)$ . Then

$$\begin{aligned} f_1'(x) &= -\frac{1}{x^2} \psi_q\left(\frac{1-x}{x}\right) \\ f_1''(x) &= \frac{2}{x^3} \psi_q\left(\frac{1-x}{x}\right) + \frac{1}{x^4} \psi_q'\left(\frac{1-x}{x}\right), \end{aligned}$$

and so,  $x^3 f_1''(x) = 2\psi_q(y) + \frac{1}{x}\psi_q'(y)$  where  $y = \frac{1-x}{x}$ . Noting that  $y < \frac{1}{2}$  and that  $\frac{1-x}{x} \geq 1$  on  $(0, \frac{1}{2}]$ , we get with the help of Lemma 1(b) that

$$x^3 f_1''(x) > 2\psi_q(y) + y\psi_q'(y) > 0.$$

It follows that  $f_1''(x) > 0$  and thus  $\Gamma_q\left(\frac{1-x}{x}\right)$  is strictly log-convex on  $(0, \frac{1}{2}]$ .

(b) Let  $f_2(x) = \log \frac{\Gamma_q(1-x)}{\Gamma_q(x)}$ . Then  $f_2''(x) = \psi_q'(1-x) - \psi_q'(x)$ . As  $1-x > x$  on  $(0, \frac{1}{2})$  and the function  $\psi_q(x)$  is strictly decreasing by Lemma 1(a), we deduce that  $f_2''(x) < 0$ . This completes the proof.  $\square$

For our next result we need the following lemma.

**Lemma 9.** *There holds*

$$\psi_q\left(\frac{1}{2}\right) < 2\psi_q(1).$$

*Proof.* By virtue of Lemma 3 applied to  $n = 2$ , we have  $\psi_q\left(\frac{1}{2}\right) = \psi_q(1) - 2\log(1+q^{\frac{1}{2}})$  and so our desired inequality means that  $\psi_q(1) > -2\log(1+q^{\frac{1}{2}})$ . Now, by a combination of Lemma 2 applied to  $x = 1$  and the relation (12), we get  $\psi_q(1) > \frac{q \log q}{1-q} - 2\log(1+q^{\frac{1}{2}})$ , which completes the proof.  $\square$

**Theorem 8.** *For any  $x \in (0, \frac{1}{2}]$ , one has*

$$\frac{\Gamma_q(1-x)}{\Gamma_q(x)} \leq \Gamma_q\left(\frac{1-x}{x}\right),$$

*with equality only for  $x = \frac{1}{2}$ .*

*Proof.* Let for any  $x \in (0, \frac{1}{2}]$ ,

$$f(x) = \log \Gamma_q\left(\frac{1-x}{x}\right) + \log \Gamma_q(x) - \log \Gamma_q(1-x).$$

As  $f\left(\frac{1}{2}\right) = 0$ , it will be enough to prove that  $f(x)$  is decreasing on  $(0, \frac{1}{2}]$ . Note first that from Theorem 7(a) we have that  $\left(\log \Gamma_q\left(\frac{1-x}{x}\right)\right)'$  is increasing on  $(0, \frac{1}{2}]$ , which implies that

$$\left(\log \Gamma_q\left(\frac{1-x}{x}\right)\right)' = -\frac{1}{x^2}\psi_q\left(\frac{1-x}{x}\right) \leq -4\psi_q(1).$$

Moreover, since the function  $\psi_q(x)$  is concave by Lemma 1(a), we have

$$\psi_q(x) + \psi_q(1-x) \leq 2\psi_q\left(\frac{x+(1-x)}{2}\right) = 2\psi_q\left(\frac{1}{2}\right).$$

Now as

$$f'(x) = \left(\log \Gamma_q\left(\frac{1-x}{x}\right)\right)' + \psi_q(x) + \psi_q(1-x),$$

we deduce from Lemma 9 and the above facts that

$$f'(x) \leq -4\psi_q(1) + 2\psi_q\left(\frac{1}{2}\right) = 2\left(\psi_q\left(\frac{1}{2}\right) - 2\psi_q(1)\right) \leq 0,$$

showing that  $f(x)$  is decreasing. This completes the proof.  $\square$

We close this section with an inequality of Ky Fan type for the  $q$ -gamma function. For Ky Fan inequalities related to the classical gamma function the reader is referred to Neuman and Sándor [15].

**Theorem 9.** For a positive integer  $k$  and  $i = 1, 2, \dots, k$ , let  $x_i \in (0, \frac{1}{2}]$  and let  $x'_i = 1 - x_i$ . Let  $A_k$  denote the arithmetic mean of  $x_i$  and let  $A'_k$  denote the arithmetic mean of  $x'_i$ . Then

$$(a) \quad \Gamma_q\left(\frac{A'_k}{A_k}\right) \leq \left(\prod_{i=1}^k \Gamma_q\left(\frac{x_i}{x'_i}\right)\right)^{\frac{1}{k}}$$

$$(b) \quad \frac{\Gamma_q(A'_k)}{\Gamma_q(A_k)} \geq \left(\prod_{i=1}^k \frac{\Gamma_q(x'_i)}{\Gamma_q(x_i)}\right)^{\frac{1}{k}}.$$

*Proof.* As the function  $\Gamma_q\left(\frac{1-x}{x}\right)$  is strictly log-convex on  $(0, \frac{1}{2}]$  by Theorem 7(a), an application of Jensen's inequality to this function yields part (a). Moreover, an application of Jensen's inequality to  $\frac{\Gamma_q(1-x)}{\Gamma_q(x)}$ , which is strictly log-concave by Theorem 7(b), gives to part (b).  $\square$

5. INEQUALITIES RELATED TO  $q$ -SERIES

**Theorem 10.** For any positive integer  $n$ , any  $x > 0$ , and any  $a \in (0, 1)$ , we have

$$(a) \quad (1 - q^x)^{1-a} < \frac{(q^{x+a}; q)_n}{(q^{x+1}; q)_n} < (1 - q^{x+a})^{1-a}$$

$$(b) \quad (1 - q^x)^{1-a} \leq {}_1\phi_0(q^{a-1}, -; q, q^{x+1}) \leq (1 - q^{x+a})^{1-a}.$$

*Proof.* Let  $f(x) = (q^x; q)_n$ . Then from  $f(x) = \sum_{i=0}^{n-1} \log(1 - q^{x+i})$ , we get

$$\left(\log f(x)\right)'' = -(\log q)^2 \sum_{i=0}^{n-1} \frac{q^{x+i}}{(1 - q^{x+i})^2} < 0$$

which means that the function  $f(x)$  is strictly log-concave and so,  $\frac{1}{f(x)}$  is strictly log-convex by Lemma 5(b). Then by Lemma 4 applied to  $\frac{1}{f(x)}$ ,

$$(1 - q^x)^{1-a} = \left(\frac{(q^x; q)_n}{(q^{x+1}; q)_n}\right)^{1-a} < \frac{(q^{x+a}; q)_n}{(q^{x+1}; q)_n} < \left(\frac{(q^{x+a}; q)_n}{(q^{x+a+1}; q)_n}\right)^{1-a} = (1 - q^{x+a})^{1-a},$$

which proves part (a). As to part (b), take limits as  $n \rightarrow \infty$  in the previous inequalities and use the  $q$ -binomial theorem 2 to obtain

$$(1 - q^x)^{1-a} \leq \frac{(q^{x+a}; q)_\infty}{(q^{x+1}; q)_\infty} = {}_1\phi_0(q^{a-1}, -; q, q^{x+1}) \leq (1 - q^{x+a})^{1-a},$$

which is the desired double inequality.  $\square$

**Theorem 11.** For any positive integer  $n$  we have

$$\frac{1}{(q; q)_\infty} \leq \inf \left\{ \frac{(q^{\frac{1}{2}}; q)_\infty^{n-1}}{(q; q^{\frac{1}{n}})_\infty (1 - q^{\frac{1}{2}})^{n-1}}, \frac{(q^{\frac{1}{2}}; q)_\infty^{n-1}}{(q^{\frac{1}{n}}; q^{\frac{1}{n}})_\infty}, \frac{(q^{\frac{1}{2}}; q)_\infty^{n-1} (-q^{\frac{1}{n}}; q^{\frac{1}{n}})_{n-1}}{(q^{\frac{1}{n}}; q^{\frac{1}{n}})_\infty (1 + q^{\frac{1}{2}})^{n-1}} \right\}.$$

*Proof.* The function  $(1 - q^x)\Gamma_q(x)$  is strictly log-convex by Askey [3]. Then by Jensen's inequality

$$\log \left( (1 - q^{(x_1 + \dots + x_k)/k}) \Gamma_q\left(\frac{x_1 + \dots + x_k}{k}\right) \right) \leq \frac{1}{k} \left( \log((1 - q^{x_1})\Gamma_q(x_1)) + \dots + \log((1 - q^{x_k})\Gamma_q(x_k)) \right),$$

which by taking  $k = n - 1$  and  $x_i = \frac{i}{n-1}$  and simplifying yield

$$\left( (1 - q^{1/2})\Gamma_q(1/2) \right)^{n-1} \leq (q^{1/n}; q^{1/n})_{n-1} \prod_{i=1}^{n-1} \Gamma_q(i/n),$$

or, by (5),

$$\left( (1 - q^{1/2})\Gamma_q(1/2) \right)^{n-1} \leq (q^{1/n}; q^{1/n})_{n-1} (\Gamma_q(1/2))^{n-1} \frac{(q^{1/2}; q^{1/2})_\infty^{n-1}}{(q; q)_\infty^{n-2} (q^{1/n}; q^{1/n})_\infty}.$$

Simplifying gives

$$(19) \quad \frac{(q^{1/2}; q)_\infty^{n-1}}{(q; q^{1/n})_\infty (1 - q^{1/2})^{n-1}} \geq \frac{1}{(q; q)_\infty}.$$

Now apply Jensen’s inequality to the strictly log-convex function  $\Gamma_q(x)$  and proceed as before to obtain

$$(20) \quad \frac{(q^{1/2}; q)_\infty^{n-1}}{(q^{1/n}; q^{1/n})_\infty} \geq \frac{1}{(q; q)_\infty}.$$

Finally, note that  $(1 + q^x)\Gamma_q(x)$  is strictly log-convex and use the same sort of argument as before to deduce that

$$(21) \quad \frac{(q^{1/2}; q)_\infty^{n-1} (-q^{1/n}; q^{1/n})_{n-1}}{(q^{1/n}; q^{1/n})_\infty (1 + q^{1/2})^{n-1}} \geq \frac{1}{(q; q)_\infty}.$$

Combining (19), (20), and (21) yields the desired result. □

**Theorem 12.** *For any integer  $n > 1$ , we have*

$$(q^{\frac{1}{2}}; q)_\infty^{\varphi(n)} \geq \sup \left\{ \prod_{d|n} (q^{\frac{1}{d}}; q^{\frac{1}{d}})_\infty^{\mu(\frac{n}{d})}, (1 - q^{\frac{1}{2}})^{\varphi(n)} \prod_{d|n} (q; q^{\frac{1}{d}})_\infty^{\mu(\frac{n}{d})}, (1 + q^{\frac{1}{2}})^{\varphi(n)} \frac{\prod_{d|n} (q^{\frac{1}{d}}; q^{\frac{1}{d}})_\infty^{\mu(\frac{n}{d})}}{\prod_{d|n} (-q^{\frac{1}{d}}; q^{\frac{1}{d}})_{d-1}^{\mu(\frac{n}{d})}} \right\}.$$

*Proof.* Note first the following well-known facts on the Euler totient function  $\varphi(n)$ :

$$\sum_{\substack{i=1 \\ (i,n)=1}}^n 1 = \varphi(n) \quad \text{and} \quad \sum_{\substack{i=1 \\ (i,n)=1}}^n i = \sum_{\substack{i=1 \\ (i,n)=1}}^n (n - i) = \frac{n\varphi(n)}{2},$$

from which it follows that

$$\sum_{\substack{i=1 \\ (i,n)=1}}^n \frac{i}{n} = \frac{\varphi(n)}{2} \quad \text{and} \quad \frac{\sum_{\substack{i=1 \\ (i,n)=1}}^n \frac{i}{n}}{\sum_{\substack{i=1 \\ (i,n)=1}}^n 1} = \frac{1}{2}.$$

Apply Jensen’s inequality to the function  $\Gamma_q(x)$  with  $k = \varphi(n)$  and  $x_i = \frac{i}{n}$  for  $i = 1, \dots, \varphi(n)$  and use the above to get

$$\log \Gamma_q\left(\frac{1}{2}\right) \leq \frac{1}{\varphi(n)} \log \prod_{\substack{i=1 \\ (i,n)=1}}^n \Gamma_q\left(\frac{i}{n}\right),$$

which by virtue of (7) means

$$\left(\Gamma_q\left(\frac{1}{2}\right)\right)^{\varphi(n)} \leq P_q(n) = \frac{\left(\Gamma_q\left(\frac{1}{2}\right)\right)^{\varphi(n)} (q^{\frac{1}{2}}; q)^{\varphi(n)}}{\prod_{d|n} \left((q^{\frac{1}{d}}; q^{\frac{1}{d}})_{\infty}\right)^{\mu\left(\frac{n}{d}\right)}}.$$

It follows that

$$(22) \quad (q^{\frac{1}{2}}; q)^{\varphi(n)} \geq \prod_{d|n} (q^{\frac{1}{d}}; q^{\frac{1}{d}})_{\infty}^{\mu\left(\frac{n}{d}\right)}.$$

In the remaining part of the proof we shall need

$$(23) \quad \prod_{\substack{i=1 \\ (i,n)=1}}^n (1 - q^{\frac{i}{n}}) = \prod_{d|n} (q^{\frac{1}{d}}; q^{\frac{1}{d}})_{d-1}^{\mu\left(\frac{n}{d}\right)}$$

which follows by applying the Möbius inversion formula to  $\prod_{i=1}^{n-1} (1 - q^{\frac{i}{n}}) = (q^{\frac{1}{n}}; q^{\frac{1}{n}})_{n-1}$ . Now apply Jensen's inequality to the function  $(1 - q^x)\Gamma_q(x)$  with  $k = \varphi(n)$  and  $x_i = \frac{i}{n}$  for  $i = 1, \dots, \varphi(n)$  to deduce

$$\log\left((1 - q^{\frac{1}{2}})\Gamma_q\left(\frac{1}{2}\right)\right) \leq \frac{1}{\varphi(n)} \log \prod_{\substack{i=1 \\ (i,n)=1}}^n (1 - q^{\frac{i}{n}})\Gamma_q\left(\frac{i}{n}\right),$$

which by virtue of (7) and (23) means

$$\left((1 - q^{\frac{1}{2}})\Gamma_q\left(\frac{1}{2}\right)\right)^{\varphi(n)} \leq \prod_{d|n} (q^{\frac{1}{d}}; q^{\frac{1}{d}})_{d-1}^{\mu\left(\frac{n}{d}\right)} \frac{\left(\Gamma_q\left(\frac{1}{2}\right)\right)^{\varphi(n)} (q^{\frac{1}{2}}; q)^{\varphi(n)}}{\prod_{d|n} \left((q^{\frac{1}{d}}; q^{\frac{1}{d}})_{\infty}\right)^{\mu\left(\frac{n}{d}\right)}}.$$

Simplifying the foregoing inequality yields

$$(24) \quad (q^{\frac{1}{2}}; q)^{\varphi(n)} \geq (1 - q^{\frac{1}{2}})^{\varphi(n)} \prod_{d|n} (q; q^{\frac{1}{d}})_{\infty}^{\mu\left(\frac{n}{d}\right)}.$$

Furthermore, apply Jensen's inequality to the function  $(1 + q^x)\Gamma_q(x)$  with  $k = \varphi(n)$  and  $x_i = \frac{i}{n}$  for  $i = 1, \dots, \varphi(n)$  and use the same argument as above to obtain

$$(25) \quad (q^{\frac{1}{2}}; q)^{\varphi(n)} \geq (1 + q^{\frac{1}{2}})^{\varphi(n)} \frac{\prod_{d|n} (q^{\frac{1}{d}}; q^{\frac{1}{d}})_{\infty}^{\mu\left(\frac{n}{d}\right)}}{\prod_{d|n} (-q^{\frac{1}{d}}; q^{\frac{1}{d}})_{d-1}^{\mu\left(\frac{n}{d}\right)}}.$$

Finally combine (22), (24), and (25) to complete the proof. □

For our next result we need the following lemma of Askey [3] which deals with the behaviour of  $\Gamma_q$  as a function of  $q$ .

**Lemma 10.** *Let  $0 < p < q < 1$ . Then*

$$\begin{aligned} (a) \quad & \Gamma_p(x) \leq \Gamma_q(x) \leq \Gamma(x), & 0 < x \leq 1 \quad \text{or} \quad x \geq 2 \\ (b) \quad & \Gamma_p(x) \geq \Gamma_q(x) \geq \Gamma(x), & 1 \leq x \leq 2. \end{aligned}$$

**Theorem 13.** *Let  $0 < p < q < 1$  and let  $n > 1$  be an integer. Then*

$$(a) \quad \frac{(p; p)_{\infty}^n}{(p^{\frac{1}{n}}; p^{\frac{1}{n}})_{\infty}} (1-p)^{\frac{n-1}{2}} \leq \frac{(q; q)_{\infty}^n}{(q^{\frac{1}{n}}; q^{\frac{1}{n}})_{\infty}} (1-q)^{\frac{n-1}{2}} \leq \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$$

$$(b) \quad \frac{(p; p)_{\infty}^n}{(p; p^{\frac{1}{n}})_{\infty}} \frac{1}{(1-p)^{\frac{n-1}{2}}} \geq \frac{(q; q)_{\infty}^n}{(q; q^{\frac{1}{n}})_{\infty}} \frac{1}{(1-q)^{\frac{n-1}{2}}} \geq \frac{(n-1)!(2\pi)^{\frac{n-1}{2}}}{n^{n-\frac{1}{2}}}.$$

*Proof.* Note first that

$$\left(\Gamma_q\left(\frac{1}{2}\right)\right)^{n-1} (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{\infty}^{n-1} = (1-q)^{\frac{n-1}{2}} (q; q)_{\infty}^{2n-2},$$

and therefore the relation (5) boils down to

$$(26) \quad \prod_{k=1}^{n-1} \Gamma_q\left(\frac{k}{n}\right) = \frac{(q; q)_{\infty}^n}{(q^{\frac{1}{n}}; q^{\frac{1}{n}})_{\infty}} (1-q)^{\frac{n-1}{2}}.$$

(a) Let  $x_i = \frac{i}{n}$  for  $i = 1, \dots, n-1$  and apply Lemma 10(a) to obtain

$$\prod_{i=1}^{n-1} \Gamma_p\left(\frac{i}{n}\right) \leq \prod_{i=1}^{n-1} \Gamma_q\left(\frac{i}{n}\right) \leq \prod_{i=1}^{n-1} \Gamma\left(\frac{i}{n}\right).$$

Now use (26) and (3) to deduce

$$\frac{(p; p)_{\infty}^n}{(p^{\frac{1}{n}}; p^{\frac{1}{n}})_{\infty}} (1-p)^{\frac{n-1}{2}} \leq \frac{(q; q)_{\infty}^n}{(q^{\frac{1}{n}}; q^{\frac{1}{n}})_{\infty}} (1-q)^{\frac{n-1}{2}} \leq \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}},$$

which is the desired inequalities.

(b) As to this part let  $x_i = 1 + \frac{i}{n}$  for  $i = 1, \dots, n-1$  and apply Lemma 10 (b) to get

$$\prod_{i=1}^{n-1} \Gamma_p\left(1 + \frac{i}{n}\right) \geq \prod_{i=1}^{n-1} \Gamma_q\left(1 + \frac{i}{n}\right) \geq \prod_{i=1}^{n-1} \Gamma\left(1 + \frac{i}{n}\right).$$

It follows by combining these inequalities with the basic facts  $\Gamma_q(x+1) = \frac{1-q^x}{1-q} \Gamma_q(x)$  and  $\Gamma(x+1) = x\Gamma(x)$  that

$$\prod_{i=1}^{n-1} \Gamma_p\left(\frac{i}{n}\right) \frac{1-p^{\frac{i}{n}}}{1-p} \geq \prod_{i=1}^{n-1} \Gamma_q\left(\frac{i}{n}\right) \frac{1-q^{\frac{i}{n}}}{1-q} \geq \prod_{i=1}^{n-1} \Gamma\left(\frac{i}{n}\right) \frac{i}{n},$$

or equivalently,

$$\frac{(p^{\frac{1}{n}}; p^{\frac{1}{n}})_{n-1}}{(1-p)^{n-1}} \frac{(p; p)_{\infty}^n}{(p^{\frac{1}{n}}; p^{\frac{1}{n}})_{\infty}} (1-p)^{\frac{n-1}{2}} \geq \frac{(q^{\frac{1}{n}}; q^{\frac{1}{n}})_{n-1}}{(1-q)^{n-1}} \frac{(q; q)_{\infty}^n}{(q^{\frac{1}{n}}; q^{\frac{1}{n}})_{\infty}} (1-q)^{\frac{n-1}{2}} \geq \frac{(n-1)!}{n^{n-1}} \prod_{i=1}^{n-1} \Gamma\left(\frac{i}{n}\right).$$

Finally, an application of (26) and (3) to the foregoing inequalities and simplifying yield

$$\frac{(p; p)_{\infty}^n}{(p; p^{\frac{1}{n}})_{\infty}} \frac{1}{(1-p)^{\frac{n-1}{2}}} \geq \frac{(q; q)_{\infty}^n}{(q; q^{\frac{1}{n}})_{\infty}} \frac{1}{(1-q)^{\frac{n-1}{2}}} \geq \frac{(n-1)!(2\pi)^{\frac{n-1}{2}}}{n^{n-\frac{1}{2}}}.$$

This completes the proof. □

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